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LETTER TO THE EDITOR

First exit time of termites and random super–normal conductor networks

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Abstract. The resistance of a random super-normal conductor network is calculated through the diffusion of a 'termite', which performs a random walk on the normal bonds and has the same probability to exit from any point on a cluster of superconducting bonds. The first time, T, that the termite exits at a distance r from the origin (averaged over many configurations and runs) is found to behave as $T \sim r^{2/k}$ for $r \ll \xi$ (ξ is the percolation correlation length) and as $T \sim r^2(p_c - p)^s$ for $r \gg \xi$ ($(p_c - p)^{-s}$ describes the divergence of the conductivity as $p \to p_c^-$, p being the concentration of the superconductor). Scaling arguments are used to show that $k = 1 + \tilde{s}/(2+\theta)$, where $(2+\theta)$ is the fractal dimension of random walks on single clusters at p_c. Numerical simulations at two dimensions ($d = 2$) yield $s = 1.34 \pm 0.10$ and $k = 1.34 \pm 0.03$, in agreement with scaling. We also show that the probability of return to the origin at time t behaves as $t^{-dk/2}$. Preliminary results at $d = 1$ and other calculational methods are also discussed.

The diffusion of a single particle on the random maze formed by a percolation process was called 'ant in a labyrinth' by de Gennes (1976). For a metal-insulator alloy, above the percolation threshold p_c of the metal, and on length scales large compared with the percolation correlation length ξ, the mean square distance travelled at time t is $\langle r^2 \rangle = Dt$, and the average diffusion coefficient D is proportional to the conductivity, $D \propto (p - p_c)^\mu \sim \xi^{-\mu/\nu}$. On smaller length scales, one encounters a slower, anomalous, diffusion (Gefen *et al* 1983). Within a single metallic cluster, the linear Fick law is replaced by

$$\langle r^2 \rangle \sim t^{2/(2+\theta)}, \tag{1}$$

and $\theta = (\mu - \beta)/\nu$ describes the decay with distance of the (single cluster) diffusion coefficient. Equation (1) associates with each time t a distance $t^{1/(2+\theta)}$. In particular, the correlation length ξ is associated with the typical time $\xi^{2+\theta}$. Many recent numerical simulations have confirmed equation (1) (see e.g. Pandey and Stauffer (1983) and references therein, or Mitescu and Roussenq (1983)) and the relation between θ and μ. Gefen *et al* also noted that although $\xi^{2+\theta}$ is the 'basic' time, equation (1) is modified when one averages over many clusters.

A complementary problem concerns the conductivity of an alloy of superconducting and normal metallic bonds (Straley 1977, 1983), which exhibits the same behaviour as the dielectric constant of a metal-insulator alloy (e.g. Wilkinson *et al* 1983), or perhaps the viscosity of a gel (de Gennes 1979). As $p \to p_c^-$ (p is the concentration of the superconductor), the average conductivity (or the dielectric constant or the viscosity)

diverges as $\sigma \sim (p_c - p)^{-s} \sim \xi^{\tilde{s}}$, where $\tilde{s} = s/\nu$. Although there exist many numerical calculations of μ, those of s 'are known as a difficult problem' (Herrmann *et al* 1984), and have only recently first been carried out using a transfer matrix algorithm (Herrmann *et al* 1984). As in the case of μ, the literature also contains some attempts to conjecture (Kertesz 1983) or to derive (Coniglio and Stanley 1984, Sahimi 1984) relations between s and other precolation exponents. Experimental measurements have also been made (Benguigui 1984).

To describe the super-normal conductor alloy, de Gennes (1980) suggested generalising the 'ant' into a 'termite', which behaves like the 'ant' on the normal bonds but which moves very quickly on superconducting clusters. de Gennes' 'stopwatch termite' moves everywhere in steps of one lattice spacing, but the clock is held still while it walks on the superconductor. An alternative 'skating termite' (Coniglio and Stanley 1983) skates in straight lines on the superconductor, i.e. it takes steps of variable length and moves in and out of the clusters. A second 'Boston termite' (Coniglio and Stanley 1984, Bunde *et al* 1985) was defined via a limiting procedure, with time steps τ_s and τ_n on the superconducting and normal bonds (and $\tau_s/\tau_n \to 0$). In the present letter we study what we call the 'Tel Aviv termite', which can exit from a superconducting cluster from *any* site with equal probabilities. The clock is held still as long as the termite is on the cluster. While we believe that this is the most realistic algorithm, it is far more difficult to actualise than any of the others mentioned above. First, it requires detailed knowledge about the characteristics of each cluster, limiting us to lattices of about 5×10^5 sites. Second, the boundary conditions become a problem, as will be discussed below.

On length scales large compared with ξ one expects the termite to obey Fick's law, $\langle r^2 \rangle = Dt$, with $D \sim \sigma \sim \xi^{\tilde{s}}$. Indeed, we present below the first numerical confirmation of this expectation. In the self-similar fractal regime, $r < \xi$, we expect the termite to move *faster* than the normal random walker, i.e. (Coniglio and Stanley 1984)

$$r^2 \sim t^k, \tag{2}$$

with $k > 1$ (in contrast to the 'ant', which has $k < 1$). Indeed, we present below scaling arguments and numerical simulations which confirm this prediction, and show that

$$k = 1 + \tilde{s}/(2 + \theta). \tag{3}$$

Instead of calculating the average of $\langle r^2 \rangle$, as appropriate for the 'ant', we argue below that it is much more appropriate to average over t, for fixed r. This corresponds to averaging $1/D$ instead of D, in analogy with averaging over the resistivity $1/\sigma$ instead of the conductivity σ. The former is more appropriate for the super-normal alloy, in which σ has a finite probability of being infinite (see also Herrmann *et al* 1984). Moreover, we believe that the relevant physical time is not the average time, t, for visiting sites at a distance r from the origin, but rather the first exit time, T, at which this distance is reached for the first time; if the electrode is placed there, the charge carrier will be 'caught' into the electrical circuit and never go back into the sample. Within a scaling theory, we expect t and T to have the same behaviour.

In the limit of infinite conductivity of the superconducting bonds we expect $D \sim \xi^{\tilde{s}}$. Following Gefen *et al*, we assume that T scales with the basic time $\xi^{2+\theta}$. Thus, we start from the scaling *ansatz*

$$D = r^2/T = \xi^{\tilde{s}} f(T/\xi^{2+\theta}). \tag{4}$$

For $x \gg 1$ we expect that $f(x) \to$ constant. For $x \ll 1$, we expect that $f(x) \propto x^{\tilde{s}/(2+\theta)}$, so that there remains no dependence on ξ (as required in this self-similar region, which should exhibit the same behaviour as at $p = p_c$). This confirms equations (2) and (3).

Using equation (2), the volume filled by sites visited after time t becomes of order $r^d \sim t^{dk/2}$. Since all these points are equally probable, the probability of return to the origin $r = 0$ is predicted to be

$$P(0, t) \sim t^{-dk/2}, \tag{5}$$

which is somewhat different from that for the 'ant' (Alexander and Orbach 1982, Rammal and Toulouse 1983).

An alternative scaling approach might attempt to replace f in equation (4) by another function, $g(r/\xi)$, with $g(x) \to$ constant for $x \gg 1$ and $g(x) \sim x^{\tilde{s}}$ for $x \ll 1$. This would yield equation (2) with

$$k = 2/(2 - \tilde{s}), \tag{6}$$

as predicted by Coniglio and Stanley (1984). As we show below, equation (6) does not agree with our numerical results. Experience in other cases (e.g. finding the average cluster size below p_c, see Stauffer (1979)) indeed shows that it is dangerous to scale r with ξ when clusters of many sizes are involved.

The above scaling theory can be similarly generalised to the case of two conductors, with conductivities $\sigma_<$ and $\sigma_>$, when scaling *à la* Straley (1977) implies

$$r^2/T = \sigma_> \xi^{-\tilde{\mu}} \bar{f}((\sigma_</\sigma_>)\xi^{\tilde{s}+\tilde{\mu}}, T/\xi^{2+\theta}), \tag{7}$$

with obvious notation and limiting behaviour.

We shall now describe details of our algorithm and present our numerical results. We start with an $L \times L$ square lattice. (Most of our results were produced with $L = 500$. Results for larger r were produced with $L = 700$.) A random fraction p of the sites is occupied by superconducting atoms, and details of the resulting clusters (e.g. their sizes) are recorded. Periodic boundary conditions were used, except when an infinite incipient cluster appeared (which would remain infinite even in the periodically continued sample). This never happened for the concentrations $p < p_c$ reported below.

The termite is 'parachuted' at random on to any site of the lattice, and then chooses to walk to one of this site's four neighbours. If the new site i is occupied by normal metal then one time step and one unit vector displacement are recorded. If the new site is occupied by a superconductor, a selection is made at random from amongst all the sites on the superconducting cluster to which this site belongs, and the termite jumps to this site, j. It then chooses one of the four neighbours of site j and if this new site is occupied by normal metal then one time step and the vector distance from site i are recorded. If it is occupied by a superconductor, another random selection from among the sites of the cluster is made, and the process continues. For the first exit time, T, we then fix a circle of radius r around the starting site, and calculate the first time at which the termite has gone beyond this circle. Since we always chose the circle to be inside our original sample, no problem with boundary conditions has to be faced.

Figure 1 shows results for T against r^2 at $p = 0.5$, 0.585 and 0.59. Each data point used some 100 ($r < 280$) or 50 ($r > 280$) different lattices, with 800 termites on each. The data in figure 1 clearly show a power law behaviour for small T and r, which we

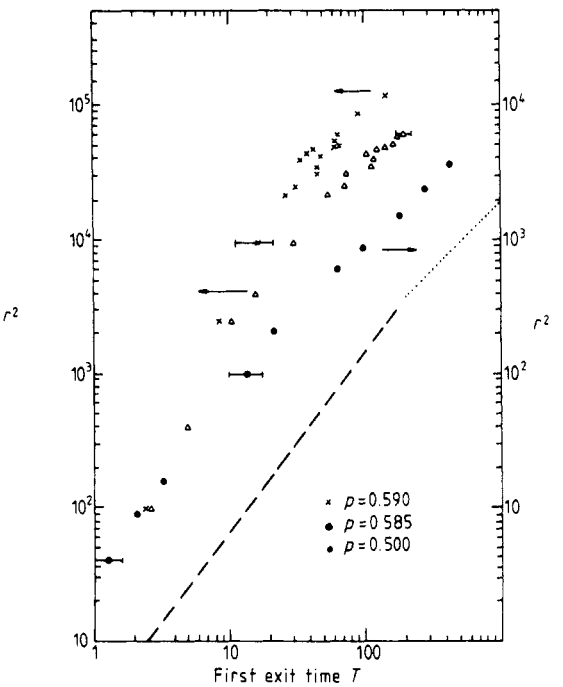

Figure 1. First exit time T as a function of r^2 for $p < p_c$ in two dimensions. Typical error bars are shown for selected points. Lines with slope $k = 1.34$ and $k = 1$ are shown to guide the eye. The left-hand scale applied to $p = 0.590$ and $p = 0.585$ and the right-hand scale applies to $p = 0.500$.

fit to equation (2), with

$$k = 1.34 \pm 0.03 \tag{8}$$

(statistical error only).

The data also show a crossover, at a radius r_x and a time T_x, to straight lines, $r^2 \sim DT$. However, it is clear from figure 1 that the data in this regime are not very accurate. This results from the fact that the first exit calculations are very time consuming, and it is difficult to collect many data for long times (large radii), especially for $p \ll p_c$.

In order to avoid these difficulties, we also performed the 'usual' ant-like calculations, in which we averaged r^2 as a function of time. Here it is essential to treat the boundaries correctly, since the termite does move back and forth through the boundaries. Whenever the termite moved through a boundary (e.g. at $x = L$) we corrected the displacement appropriately (e.g. to $x_j - X_i + L$). Some of our results (samples with $L = 500$, with each point representing 60 lattices and 800 termites) are shown in figure 2. The data clearly show a Fick law, $\langle r^2 \rangle = Dt$, for $r > r_x$. The data from figure 1, in this regime, fall on similar (although less accurate) lines. The values of r_x are consistent with $1.2(p_c - p)^{-\nu}$, $\nu = \frac{4}{3}$, which is an estimate of ξ (Kapitulnik *et al* (1983), using also the amplitude ratio $\xi_+/\xi_- \simeq 1.5$ (Aharony 1980)). The values of T_x are also not inconsistent with $r_x^{2+\theta}$, using $\theta = (\mu - \beta)/\nu \simeq 0.87$ (Herrmann *et al* 1984, Zabolitsky 1984, Hong *et al* 1984, Lobb and Frank 1984). The slopes, D, are plotted against

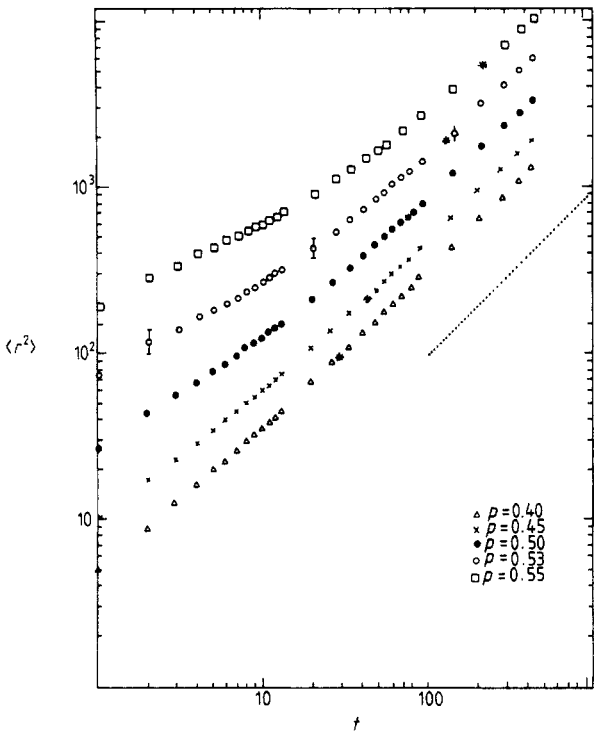

Figure 2. Mean square displacement $\langle r^2 \rangle$ as a function of t for $p = 0.40$, 0.45, 0.50, 0.53 and 0.55. The points where $\Gamma = 1.2(p_c - p)^{-4/3}$ are denoted by asterisks. A line of slope $k = 1$ is drawn to guide the eye, and typical error bars are shown for $p = 0.50$.

$(p_c - p)$ in figure 3, and they clearly exhibit the power law $D \sim (p_c - p)^{-s}$, with $s = 1.34 \pm 0.1$ (where we use $p_c = 0.592\,77$ (Gebele 1984)). Using $\nu = \frac{4}{3}$, this value of s is consistent with the (more accurate) values available from other sources (e.g. Herrmann *et al* 1984). It is satisfactory to note that this value of s (or of $\tilde{s} = s/\nu$) is consistent

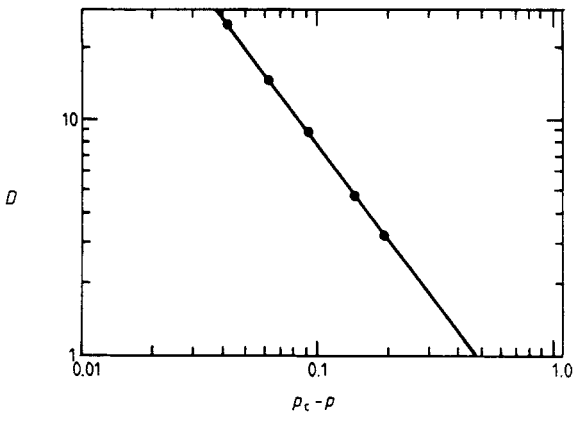

Figure 3. The asymptotic value of $D = \langle r^2 \rangle / t$ as a function of $(p_c - p)$ for $p = 0.40$, 0.45, 0.50, 0.53 and 0.55. The slope of the line is $s = 1.34 \pm 0.10$.

with our result (8), using equation (3). In contrast, our k and $\tilde{s}$ do not obey equation (6).

We note that the data of figure 2 at small values of r do not look like those of figure 1: the values of $\langle r^2 \rangle$ are too large, and there is no single power law which can describe them. We attribute this to the high sensitivity of the calculation to the type of average used. When we averaged $1/r^2$, instead of r^2 (eliminating the value $r=0$), the resulting $1/\langle 1/r^2 \rangle$ was lower. We believe that this indicates that the correct average should be $\langle t \rangle$, as used in the first part of this report. The sensitivity disappears for $r \gg r_x$, where the system behaves as a homogeneous dirty material.

As noted above, an alternative estimate of k can be made from the study of the probability of return to the origin, $P(0, t)$. Data for $P(0, t)$ can be extracted from the calculation of $\langle r^2 \rangle$ as a function of time simply by recording the number of times a termite returns to the origin, at each time step. These data oscillate for the first few steps; while return to the origin after one (or three) steps is not forbidden, as in a usual random walk, it is less likely than return after two (or four) steps. Figure 4 illustrates $P(0, t)$ for $p=0.5$, and we deduce the estimate $k = 1.3 \pm 0.3$.

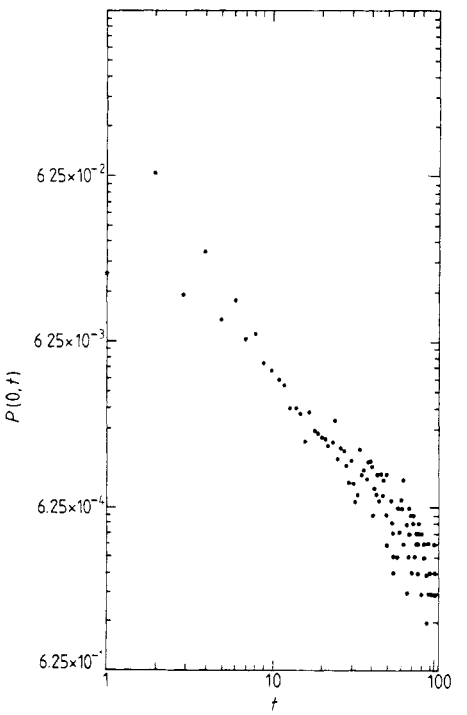

Figure 4. Probability of return to the origin after t time steps for $p=0.53$.

Finally, we present some preliminary results for one dimension. Here one expects $s=\nu=1$, $t=\beta=0$ (e.g. Coniglio and Stanley 1984), so that we predict $k=\frac{3}{2}$. Indeed, figure 5 presents our data for T, and we find $k = 1.47 \pm 0.03$, in agreement with equation (3) and not with equation (6). As for $d=2$, the data for large r are not yet in the asymptotic regime, so that one cannot yet obtain s from $D = r^2/T$.

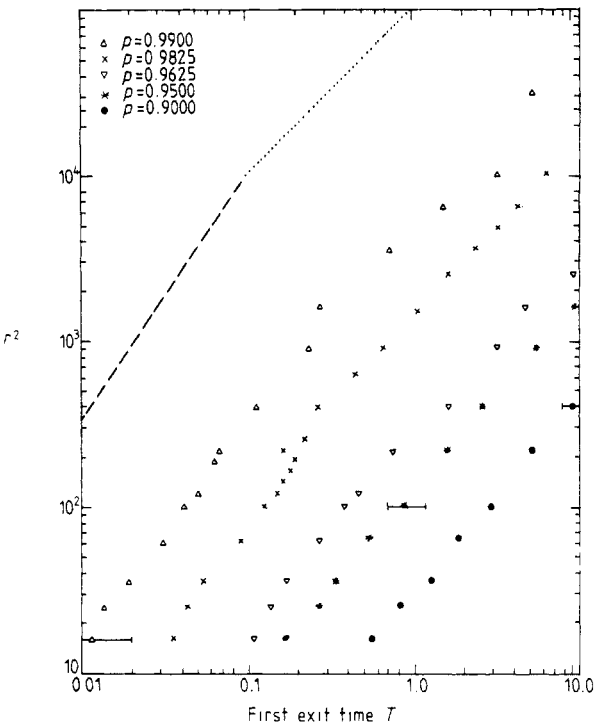

Figure 5. First exit time T as a function of r^2 for $p < p_c$ in one dimension. Typical error bars are shown for selected points. Lines with slope $k = 1.47$ and $k = 1$ are drawn to guide the eyes, but the curves for $p = 0.95$ and $p = 0.9$ appear to have $k = 1$ for $r^2 > 10$.

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Note added in proof. Further study is needed on the relations between different averages, universality classes, and remote species (Coniglio and Stanley 1984, Bunde *et al* 1984, Pandey 1984, Sahimi 1985).

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